Multi-Domain Solutions of PDEs Posed on Perforated Domains

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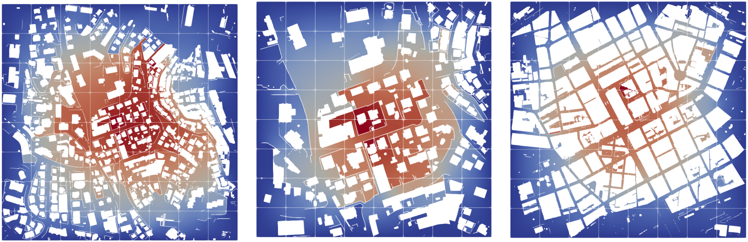
Outline

Motivation

- Efficiently solve problems on perforated domains.
  - Numerous holes representing buildings and walls in urban data;
  - Can be considered a heterogeneous domain with coefficients $0, 1$.
  - Expect corner singularities
  - Want to avoid global fine-scale solve.
- We begin with the linear Poisson equation before moving to nonlinear problems (Diffusive Wave model).
- Applications: flood modelling in urban areas.



Model PDE: Linear

- D: Open simply connected polygonal domain in $\mathbb{R}^2$;
- $(\Omega_{S,k})_k$: Finite family of perforations in D;
- $\Omega_S = \bigcup_k \Omega_{S,k}$ and $\Omega = D \setminus \overline{\Omega_S}$.

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_S, \\ u = 0 & \text{on} \quad \partial\Omega \setminus \partial\Omega_S. \end{cases}$$

With a P1 finite element discretization, this discretely becomes the linear system

$$\mathbf{Au} = \mathbf{f}.$$

Domain Decomposition Approach

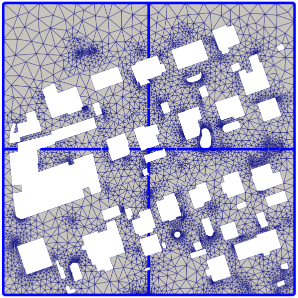
- 'Divide and conquer': Break up problem into subdomains;
- Two levels of discretization: 'Coarse' and 'fine';
- Local subdomain solves can be done in parallel;
- Can use overlapping Schwarz methods as iterative solver or as preconditioner for Krylov;

Idea: Solve model problem on each subdomain locally, with boundary conditions taken from adjacent subdomains when possible.

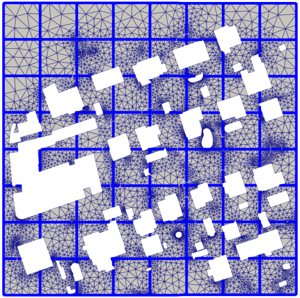
Coarse-cell conforming triangulation

Mesh generation process:

- Larger $N \rightarrowtail$ more basis functions, larger coarse matrix ;
- Triangulate after nonoverlapping coarse cell partitioning Ω'_j;
- Overlap subdomains by layers of triangles for RAS.



2×2 subdomains

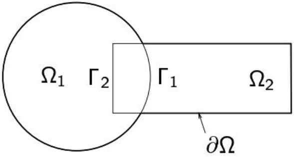


8×8 subdomains

Alternating Schwarz Introduction for $\mathcal{L}u = f$: 2 subdomains

Continuously, the Schwarz iteration is given by

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f && \text{in} \quad \Omega_1 && \mathcal{L}u_2^{n+1} = f && \text{in} \quad \Omega_2 \\ u_1^{n+1} &= u_2^n && \text{on} \quad \Gamma_1 && u_2^{n+1} = u_1^{n+1} && \text{on} \quad \Gamma_2 \end{aligned}$$



- $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$.
- Solve on Ω_1, use information from Ω_1 as boundary condition for the solve on Ω_2, etc.

Parallel Schwarz Introduction for $\mathcal{L}u = f$: 2 subdomains

Continuously, the local classical additive Schwarz iteration is given by

$$\begin{aligned} \mathcal{L}u_1^{n+1} &= f \quad &\text{in} \quad &\Omega_1 \qquad & \mathcal{L}u_2^{n+1} &= f \quad &\text{in} \quad &\Omega_2 \\ u_1^{n+1} &= u_2^n \quad &\text{on} \quad &\partial\Omega_1 \cap \Omega_2 \qquad & u_2^{n+1} &= u_1^n \quad &\text{on} \quad &\partial\Omega_2 \cap \Omega_1 \end{aligned}$$

Extending the iteration to multiple subdomains, the algorithm is given by the following:

$$\begin{aligned} \mathcal{L}u_j^{n+1} &= f \quad &\text{in} \quad &\Omega_i \\ u_j^{n+1} &= u_i^n \quad &\text{on} \quad &\partial\Omega_j \cap \Omega_i \end{aligned}$$

for $j = 1, \ldots, N$ and j such that $\partial\Omega_i \cap \Omega_j$ is non-empty.

- At each iteration, use information from adjacent subdomains at **previous** iteration $\rightarrow$ parallel iteration.

Algebraic Form

Algebraically, the global stationary (RAS) iteration becomes

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \left(\sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) (\mathbf{f} - \mathbf{A}\mathbf{u}^n)$$

and the preconditioned system is given by

$$\left(\sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) \mathbf{A}\mathbf{u} = \left(\sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j \right) \mathbf{f}$$

- $\mathbf{R}_j$: Boolean restriction matrices for Ω_j;
- $\mathbf{D}_j$: Partition of unity matrices (deal with overlap);
- $\mathbf{R}_j$ notation allows for global iteration, algebraic definition, overlapping subdomains.

1D example- Restriction, partition of unity matrices

Given set of indices $\mathcal{N} = \{0, 1, 2, 3, 4\}$: partitioned into $\mathcal{N}_1 = \{0, 1, 2, 3\}$ and $\mathcal{N}_2 = \{2, 3, 4\}$, restriction and partition of unity matrices are given as

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

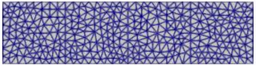
- Satisfies $\mathbf{I} = \sum_{j=1}^{2} \mathbf{R}_j^T \mathbf{D}_j \mathbf{R}_j$.

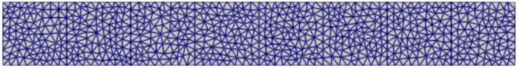
Need for coarse correction

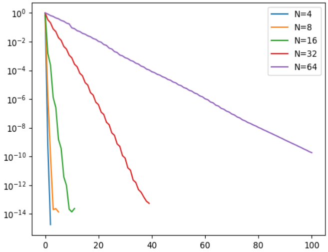
- Coarse corrections allows for global communication between all subdomains.
- Coarse correction (two-level methods) necessary for scalability for large number of subdomains.
- Generally, without coarse correction: Iterations scale with N.

Numerical Comparison: Without coarse correction

- Weak Scalability: fixed subdomain and fine triangulation size, keep $\frac{H}{h}$ constant.
- Shown on homogeneous 2D domain (subdomains in 1 dimension).







(Some) existing overlapping Schwarz coarse spaces

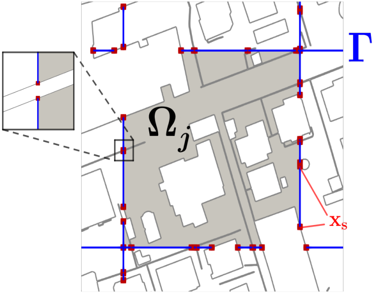
- Nicolaides: Piecewise constant by subdomain;
- Spectral spaces (eigenvalue problems): DtN, GenEO, SHEM (spectrally enriched MSFEM);
- Energy-minimizing spaces: GDSW, AGDSW, RGDSW;
- Multi-scale FEM: MsFEM
  - Numerically compute harmonic basis functions.
  - Used to approximate solution on coarse grid, but can use as DD coarse space!

Choice of coarse space

- Idea: want to take advantage of a-priori location of perforations (buildings/walls);
- Want robustness with respect to perforation size/location (even along subdomain interfaces);
- Want to choose a coarse space with approximation properties to improve convergence;
- Choose: Local harmonic basis functions occuring at intersection of a perforation with the coarse skeleton.
  - Think of as 'enriching' MsFEM coarse space.
  - Based on nonoverlapping subdomains.

Coarse grid nodes for coarse space basis functions

- Nonoverlapping skeleton: $\Gamma = \bigcup_{j\in\{1,\ldots,N\}} \partial\Omega'_j$;
- $(e_k)_{k=1,\ldots,N_e}$: Partitioning of Γ;
  - each "coarse edge" e_k is an open planar segment;
- Set of coarse grid nodes: $\bigcup_{k=1,\ldots,N_e} \partial e_k$
- $(\phi_s)_{s\in\{1,\ldots,N_x\}}$: Locally harmonic basis functions for each coarse grid node.
- $\#$ of coarse grid nodes is automatically generated.

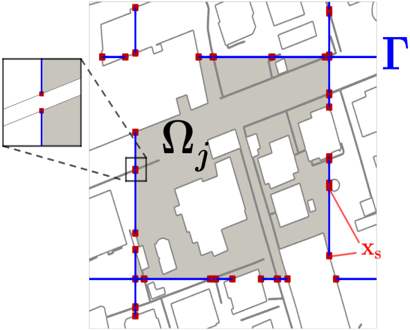


Basis functions: boundary conditions

For each coarse grid node $\mathbf{x}_s$, define $g_s : \Gamma \rightarrow [0,1]$ as: for $i = 1, \ldots N_{\mathbf{x}}$,

$$g_s(\mathbf{x}_i) = \begin{cases} 1, & s = i, \\ 0, & s \neq i, \end{cases}$$

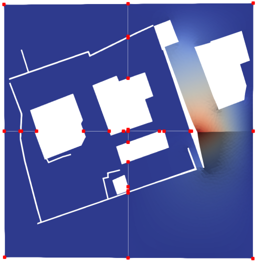
- g_s is linearly extended on the remainder of Γ.
- Can also include higher-order polynomials on coarse edges.



Basis functions: Harmonic local solutions

For all nonoverlapping $\left(\Omega'_j\right)_{j\in\{1,\dots,N\}}$ and $s = 1, \dots, N_{\mathbf{x}}$, to obtain $\phi_{s,j} = \phi_s|_{\Omega_j}$, solve

$$\begin{cases} -\Delta\phi_{s,j} = 0 & \text{in} \quad \Omega'_j, \\ -\dfrac{\partial\phi_{s,j}}{\partial n} = 0 & \text{on} \quad \partial\Omega'_j \cap \partial\Omega_S, \\ \phi_{s,j} = g_s & \text{on} \quad \partial\Omega'_j \setminus \partial\Omega_S. \end{cases}$$



- $\mathrm{supp}(\phi_s) = \{\bigcup_j \Omega'_j \mid \mathbf{x}_s$ is a coarse grid node belonging to $\partial\Omega'_j\}$.
- Continuously, the coarse space is given by $V_H = \mathrm{span}\{\phi_s\}$.
- Discretely, columns of coarse matrix $\mathbf{R}_0^T$ are the discrete harmonic basis functions.

2-level RAS iteration: N Subdomains

Combine (multiplicitavely) the 1-level RAS iteration

$$M_{RAS,1}^{-1} = \sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j$$

with the coarse approximation

$$M_0^{-1} = \mathbf{R}_0^T (\mathbf{R}_0 \mathbf{A} \mathbf{R}_0^T)^{-1} \mathbf{R}_0.$$

and solve

$$\mathbf{u}^{n+\frac{1}{2}} = \mathbf{u}^n + M_{RAS,1}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^n),$$
$$\mathbf{u}^{n+1} = \mathbf{u}^{n+\frac{1}{2}} + M_0^{-1}(\mathbf{f} - \mathbf{A}\mathbf{u}^{n+\frac{1}{2}}),$$

- $\mathbf{R}_j$: Correspond to overlapping subdomains.

The 2-level preconditioner for Krylov

Combine (additively) the 1-level RAS iteration

$$M_{RAS,1}^{-1} = \sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{D}_j (\mathbf{R}_j \mathbf{A} \mathbf{R}_j^T)^{-1} \mathbf{R}_j$$

with the coarse approximation

$$M_0^{-1} = \mathbf{R}_0^T (\mathbf{R}_0 \mathbf{A} \mathbf{R}_0^T)^{-1} \mathbf{R}_0.$$

to give

$$M_{RAS,2}^{-1} = M_0^{-1} + M_{RAS,1}^{-1}.$$

and solve

$$M_{RAS,2}^{-1} \mathbf{A} \mathbf{u} = M_{RAS2}^{-1} \mathbf{f}.$$

Approximation properties: Multiscale approximation

Discretely, given

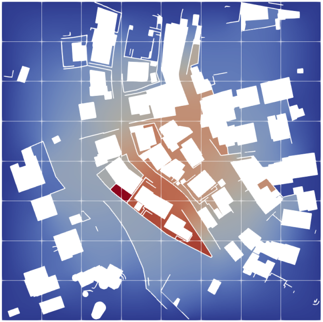
$$M_0^{-1} = \mathbf{R}_0^T(\mathbf{R}_0\mathbf{A}\mathbf{R}_0^T)^{-1}\mathbf{R}_0.$$

the coarse approximation is the solution of

$$\mathbf{u}_H = M_0^{-1}\mathbf{f}.$$

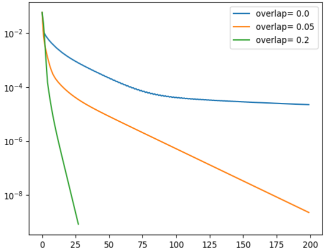
- Can use u_H as initial iterate for iteration, Krylov methods.

Linear Numerical Results: Iterative+Krylov, real data set

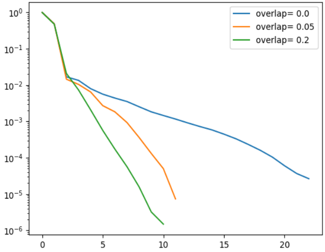


- Compare iterative to Krylov with various overlap values;
- Multiple singularities and no analytical solution available.

Numerical Results: Iterative RAS (Real data)



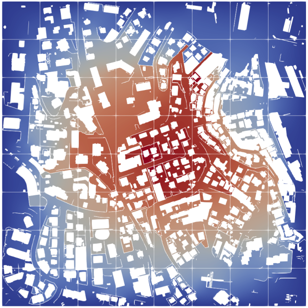
Iterative



Krylov

- Fast convergence with Krylov acceleration.
- As expected, faster convergence with larger overlap.

Experiment 3: Krylov Scalability, large real data set



$\approx$ 300K DOFS in FE triangulation.

- Want to show scalability:
- "Strong" scalability tests: Keep model domain and h constant, vary N.

Numerical Results: Krylov (table)

	Trefftz		
	it.		dim. (rel)
N	min	$\frac{H}{20}$	
16	56	22	400 (16.0)
64	56	26	880 (10.9)
256	59	30	1912 (6.6)
1024	61	28	4253 (3.9)

- Relative dimension (rel): Compared to would-be homogeneous domain, $\frac{\dim(R_0)}{(\sqrt{N}+1)^2}$.
- Relative dimension reduces as N increases;
- Trefftz-like space produces scalable, accelerated iterations.

Nonlinear Problem: Diffusive Wave model

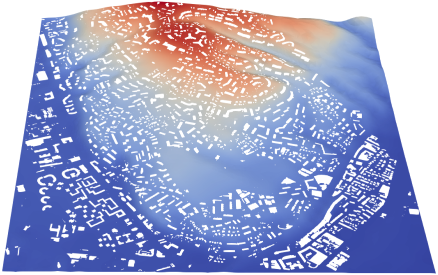
$$\left\{ \begin{array}{rcllll} \partial_t u + \mathrm{div}\mathcal{F}(x,u,\nabla u) & = & f & \text{in} & \Omega, & \\ \mathcal{F}(u)\cdot \mathbf{n} & = & 0 & \text{on} & \text{on} & \partial\Omega \cap \partial\Omega_S, \\ u & = & g & \partial\Omega \setminus \partial\Omega_S. & & \end{array} \right.$$

$$\mathcal{F}(x,u,\nabla u) = h(u, z_b(\mathbf{x}))^{\alpha} ||\nabla u||^{p-2} \nabla u,$$

- $z_b(\mathbf{x})$: Bathymetry;
- $h(u, z_b(\mathbf{x})) = \max(u - z_b(\mathbf{x}), 0)$: Water depth;
- κ: Friction coefficient;
- $\alpha > 1, 1 < p \leq 2$.

Forming Realistic Problem

- Realistic bathymetry /topography of Nice, France: 5m data;
- Rainfall data (source term): Can be taken from previous flood events (rain gauge data);
- Discretization of Problem: FEM/FV Hybrid with upwinding.



Discretization

We obtain the nonlinear system

$$F(U^n) = \frac{1}{\Delta t} M(U^n - U^{n-1}) + K(U^n) = 0, \quad (1)$$

where M is the (lumped) mass-matrix.

- Time derivative is computed via backward-Euler;
- $K(U^n)$ is discretization of nonlinear term (FEM/FVM);
- Perform upwinding on $h(u, z_b(\mathbf{x}))^\alpha$ term (due to degeneracy);
- Adaptive time-stepping may be necessary for Newton's method.

Nonlinear Preconditioning

Goal: instead of $F(U) = 0$, solve $N(F(U)) = 0$.

- $N(v) = 0 \rightarrow v = 0$;
- $N(F(v))$ straightforward to compute.

Recall from linear problem $\rightarrow$ fixed point iteration leads to a well-suited preconditioner.
Idea: From some fixed point iteration

$$U^{n+1} = P(U^n), \tag{2}$$

solve $\mathcal{F}(U) = P(U) - U = 0$ via nonlinear solve.

- $\mathcal{F}(U) = 0$ is preconditioned nonlinear system.

Nonlinear RAS iteration

Similarly to the linear problem, use local subdomain solves and glue together to form fixed-point iteration.

$$U^{n+1} = \sum_j R_j^T D_j G_j(U^n), \tag{3}$$

where $G_j(U^n)$ is the solution of

$$R_j F(R_j^T G_j(U^n) + (I - R_j^T R_j)U^n)) = 0. \tag{4}$$

- Local subproblems are solved via Newton with negligible cost;
- Local solves can be done in parallel.

RASPEN

As mentioned, solve $\mathcal{F}(U) = \sum_j R_j^T D_j G_j(U) - U = 0$ (RASPEN). via Newton.

- an "improvement" from ASPIN, converges in the overlap;
- Inner nonlinear solves allow for computation of exact Jacobian $\nabla\mathcal{F}$, or specifically the matrix-vector product $\nabla\mathcal{F}v$ for some v.

RASPEN: Computation of Jacobian

Recall equation for local nonlinear solves:

$$R_j F(R_j^T G_j(U^n) + (I - R_j^T R_j)U^n)) = 0.$$

Taking the derivative of this equation, we obtain

$$\nabla G_j(U^n) = R_j - [R_j \nabla F(U^n) R_j^T]^{-1} R_j \nabla F(U^n);$$

This gives

$$\begin{aligned} \nabla \mathcal{F}(U^n) &= \nabla(U^n - \sum_j R_j^T D_k \nabla G_j(U^n)) \\ &= \sum_j R_j^T D_j [R_j \nabla F(U^n) R_j^T]^{-1} R_j \nabla F(U^n) \end{aligned}$$

- $R_j \nabla F(U^n) R_j^T$, $\nabla F(U^n)$ can be reused from local nonlinear solves.

One-level RASPEN

The algorithm, for each time step, is given by: For outer iteration $n = 0, \dots,$ to convergence,

- Solve $\widehat{U}^n = \sum_j R_j^T D_j G_j(U^n)$ by gluing local solutions;
- Set $\mathcal{F}(U) = U - \widehat{U}^n$;
- Solve $U^{n+1} = U^n - [\nabla\mathcal{F}(U^n)]^{-1}\mathcal{F}(U^n)$ via GMRES, where $\nabla\mathcal{F}(U^n)$ is assembled as a linear operator.

Two-level RASPEN

While there are many different ways to choose the coarse correction (including FAS !cite inspired by Multigrid, we add the coarse correction multiplicitavely, with a discrete matrix R_0.

The algorithm, for each time step, given by: For outer iteration $n = 0, \ldots,$ to convergence,

- solve local subproblems $R_j F(R_j^T G_j(U^n) + (I - R_j^T R_j)U^n)) = 0$ for $G_j(U^n)$;
- Set $\widehat{U}^n = \sum_j R_j^T D_j G_j(U^n)$ by gluing local solutions;
- Solve coarse problem $R_0 F(\widehat{U}^n - R_0^T c_o^n) = 0$ for c_0^n;
- Set $\mathcal{F}(U^n) = U^n - \widehat{U}^n + R_0^T c_o^n$;
- Solve $U^{n+1} = U^n - [\nabla \mathcal{F}(U^n)]^{-1} \mathcal{F}(U^n)$ via GMRES, where $\nabla \mathcal{F}(U^n)$ is assembled as a linear operator.

Coarse Galerkin Approximation

Coarse Galerkin Formulation: solve

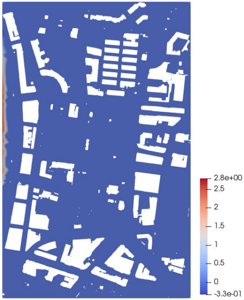
$$\mathbf{R}_0 F(\mathbf{R}_0^T u_H^n) = 0 \tag{5}$$

for each time step via Newton.

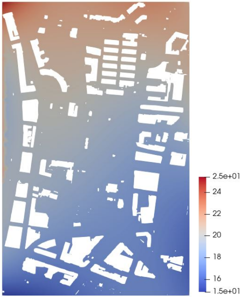
- Residual still takes global vector as input, but input vector is sparse;
- Much more efficient than global Newton solve (cheaper outer iterations).

Setup example model problem

- Excessive water flow coming from Paillon river in Nice, France;
- Dirichlet boundary conditions with initial condition $u_0 > z_b$ at leftmost boundary (river).
- $\alpha = \frac{3}{2}, p = 2$ (ignoring gradient term), 0 source term.

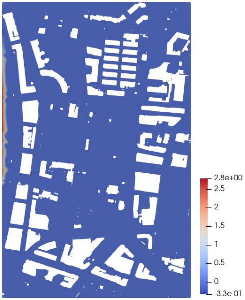


initial h

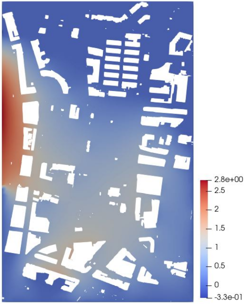


initial u

Solution at final time (h)



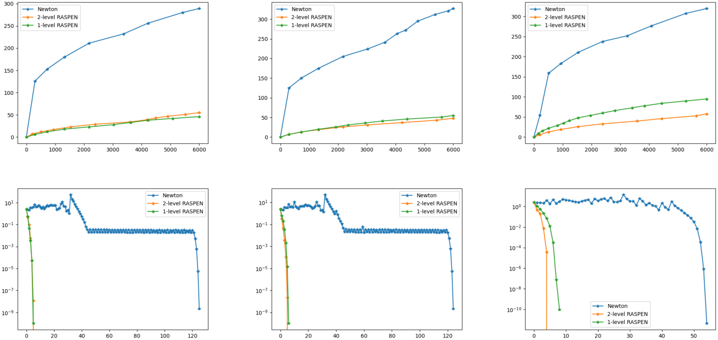
$t = 0$



$t = t_f$

- Effect of z_b is visible.

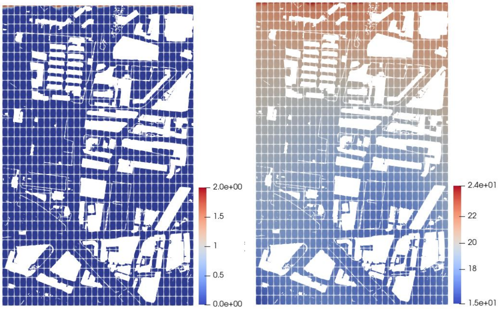
Numerical results

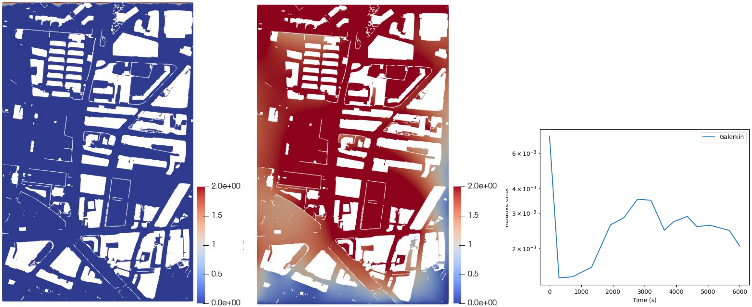


- Left to right: N=2, 4,8.
- Top: cumulative iterations over time. Bottom: convergence history for first time step.
- $\mathrm{dt}_0 = 5$ minutes, with increasing/decreasing by $\sqrt{2}$ depending on convergence.

Setup example model problem: Comparison of Coarse Galerkin and Newton

- Excessive water flow coming from the top of the domain with Dirichlet boundary conditions;
- $\alpha = \frac{3}{2}, p = 2$ (ignoring gradient term).
- Comparison between Coarse Galerkin and Newton.





- Left to right: solution h at initial time, solution h at final time, error between coarse Galerkin and Newton over time.
- Runtimes are 370 seconds (coarse Galerkin), 3614 seconds (Newton).
- Coarse Galerkin method gives acccurate solution

Closing Remarks

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- We have presented a novel Trefftz coarse space that can be used to approximate the fine-scale solution;
- The space can also be used in combination with Schwarz methods to achieve fine-scale accuracy.
- For the nonlinear problem, nonlinear preconditioning can be used in a similar manner to Krylov acceleration (accelerating a fixed-point iteration);
- Performing the coarse Galerkin method is cheap, easy to implement, and reasonably accurate.

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